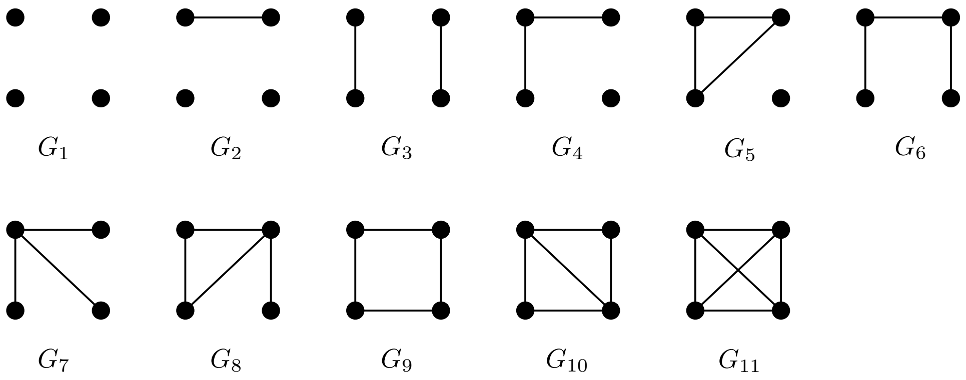
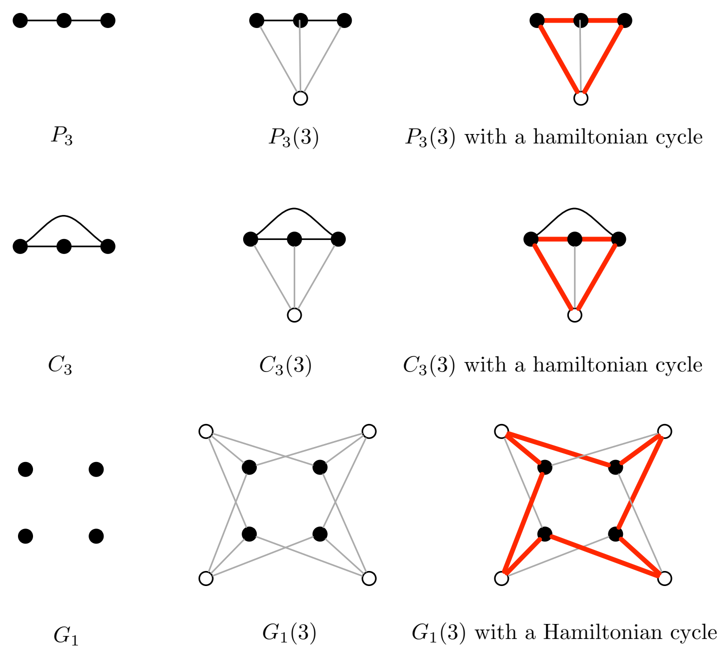
▶ Exercise 10-4

For a graph $G = (V, E)$ of order $n \geqslant 3$, then the graph $G(3)$ is obtained from G by adding a new vertex v_S for each 3-element subset S of V and joining v_S to each vertex in S. Find all such graphs G for which $G(3)$ is Hamiltonian.

Solution. We claim that $G(3)$ is Hamiltonian if and only if G has order 4 (i.e., G is a 4-vertices graph) or G is isomorphic to P_3 or C_3. We first observe that all non-isomorphic 4-vertices graphs are shown below.



We can easily check that $P_3(3)$, $C_3(3)$ and $G_1(3)$ are Hamiltonian by the following drawing (where the RED lines indicate a Hamiltonian cycle).



Since every graph $G_i(3)$ for $2 \leqslant i \leqslant 11$ contains $G_1(3)$ as a subgraph, the hamiltonicity of $G_1(3)$ implies that $G_i(3)$ is also Hamiltonian.

Next, we will show that $G(3)$ is not a Hamiltonian graph if G has order $n \geqslant 5$. So let $n \geqslant 5$ and G be a graph of order n. Let $W = \{v_S \,|\, S \subseteq V \text{ and } |S| = 3\}$ and

$G' = G(3) - V$ (i.e., the removal of all vertices of V from $G(3)$). Clearly, the number of components in G' is $k(G') = |W| = \binom{n}{3}$. From the result of Exercise 10-1, we have known that if $k(G(3) - V) > |V|$ (that is, $\binom{n}{3} > n$), then $G(3)$ is not Hamiltonian. For a positive integer n, the inequality $\binom{n}{3} > n$ is equivalent to $n > 4$. That is, for $n \geqslant 0$,

$$\binom{n}{3} > n \Leftrightarrow \frac{n(n-1)(n-2)}{6} > n \Leftrightarrow n^3 - 3n^2 - 4n > 0 \Leftrightarrow n(n+1)(n-4) > 0 \Leftrightarrow n > 4.$$

Since $n \geqslant 5$, the graph $G(3)$ is not Hamiltonian. □