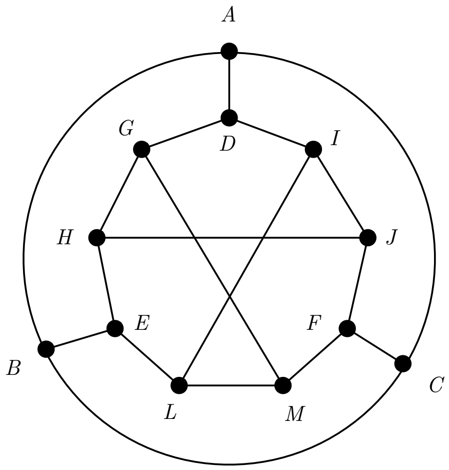
### ▶ Exercise 10-5

Determine whether the following graph is Hamiltonian.




**Solution.** Let $\mathcal{G}$ be the graph shown above, and we will prove that $\mathcal{G}$ is not hamiltonian. Suppose to the contrary that $\mathcal{H}$ is a Hamiltonian cycle of $\mathcal{G}$. Then $\mathcal{H}$ must contain at least one of the three edges $AD$, $BE$, and $CF$ connecting the outer to the inner vertices. Since the graph is symmetric, there is no loss of generality in assuming that $AD \in \mathcal{H}$. Clearly, precisely one of the two edges $AB$ and $AC$ is in $\mathcal{H}$, and precisely one of the two edges $DG$ and $DI$ is in $\mathcal{H}$. Again, by symmetry, we consider the following two cases.

CASE 1: $AB, DG \in \mathcal{H}$. In this case, $AC, DI \notin \mathcal{H}$. Since $AC \notin \mathcal{H}$, it implies $CB, CF \in \mathcal{H}$. Also, since $DI \notin \mathcal{H}$, it implies $IJ, IL \in \mathcal{H}$ (see Figure (a) for illustration). Now, because precisely two edges incident with $B$ are in $\mathcal{H}$, $BE \notin \mathcal{H}$ and thus $EH, EL \in \mathcal{H}$. Again, because precisely two edges incident with $L$ are in $\mathcal{H}$, $LM \notin \mathcal{H}$ and thus $MG, MF \in \mathcal{H}$ (see Figure (b) for illustration). At this point, however, $\mathcal{H}$ contains the proper cycle $ABCFMGDA$, a contradiction.

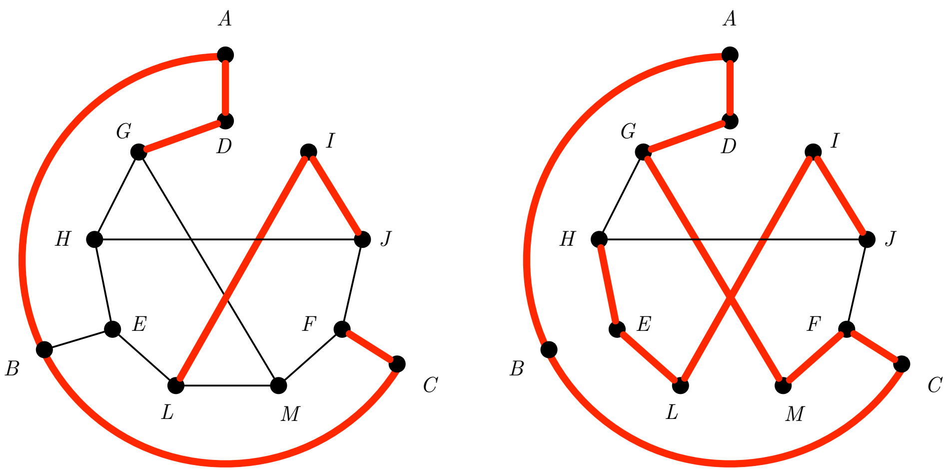



Figure (a) Figure (b)

Case 2: $AB, DI \in \mathcal{H}$. In this case, $AC, DG \notin \mathcal{H}$. Since $AC \notin \mathcal{H}$, it implies $CB, CF \in \mathcal{H}$. Also, since $DG \notin \mathcal{H}$, it implies $GH, GM \in \mathcal{H}$ (see Figure (c) for illustration). Now, because precisely two edges incident with $B$ are in $\mathcal{H}$, $BE \notin \mathcal{H}$ and thus $EH, EL \in \mathcal{H}$. Again, because precisely two edges incident with $H$ are in $\mathcal{H}$, $HJ \notin \mathcal{H}$ and thus $JI, JF \in \mathcal{H}$ (see Figure (b) for illustration). At this point, however, $\mathcal{H}$ contains the proper cycle $ABCFJIDA$, a contradiction.

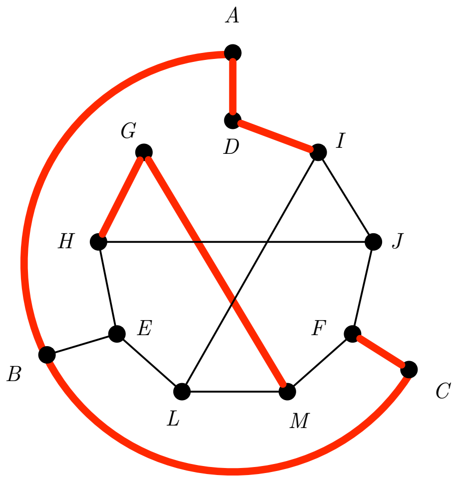


Figure (c)

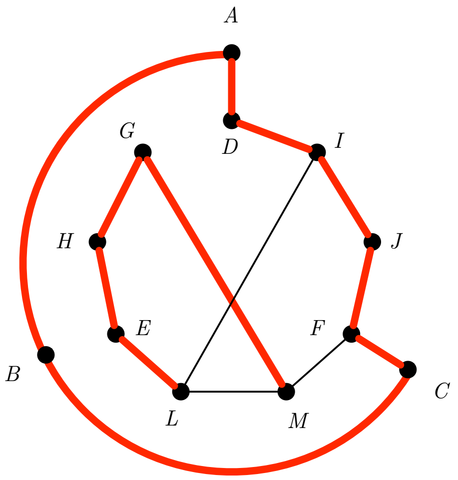


Figure (d)

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