▶ **Exercise 9-4**

The degree of every vertex of a graph G of order $2n + 1 \geqslant 5$ is either $n + 1$ or $n + 2$. Prove that G contains at least $n + 1$ vertices of degree $n + 2$ or at least $n + 2$ vertices of degree $n + 1$.

Proof. Let x be the number of vertices of degree $n+1$ in G, and let y be the number of vertices of degree $n + 2$ in G. Then, $x + y = 2n + 1 \geqslant 5$. Suppose to the contrary that G contains at most n vertices of degree $n + 2$ (i.e., $y \leqslant n$) and at most $n + 1$ vertices of degree $n + 1$ (i.e., $x \leqslant n + 1$). We now consider the following two cases:

Case 1: n is odd. Since $n+2$ is odd and the number of odd vertices is even by Euler formula, y must be even. In this case, since $y \leqslant n$ and n is odd, it implies that y cannot more than $n - 1$ (i.e., $y \leqslant n - 1$). Thus, $x + y \leqslant (n + 1) + (n - 1) = 2n$, a contradiction.

Case 2: n is even. Since $n+1$ is odd and the number of odd vertices is even by Euler formula, x must be even. In this case, since $x \leqslant n + 1$ and $n + 1$ is odd, it implies that x cannot more than n (i.e., $x \leqslant n$). Thus, $x + y \leqslant n + n = 2n$, a contradiction. □