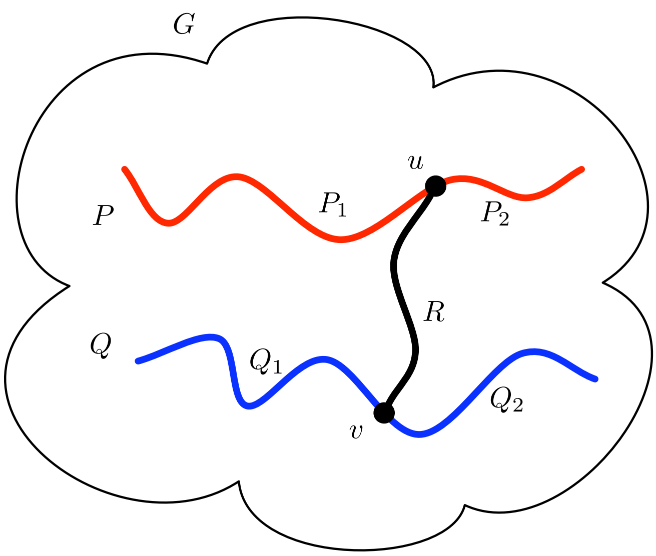
▶ **Exercise 9-6**

Prove that if P and Q are two longest paths in a connected graph, then P and Q have at least one vertex in common.

Proof. Assume, to the contrary, that there exists a connected graph G containing two longest paths P and Q that have no vertex in common. That is, $|P| = |Q|$ and $P \cap Q = \emptyset$. Since G is connected, there exists a path R joining two vertices u and v, where u is on P, v is on Q, and no interior vertex of R belongs to P and Q. Suppose $P = P_1uP_2$ is divided into two subpaths P_1 and P_2 such that $P_1 \cap P_2 = \{u\}$, and $Q = Q_1vQ_2$ is divided into two subpaths Q_1 and Q_2 such that $Q_1 \cap Q_2 = \{v\}$. See the following figure for illustration.



Without loss of generality, we assume that $|P_1| \geqslant |P_2|$ and $|Q_1| \geqslant |Q_2|$. Obviously, we can find a path P_1uRvQ_1 which has length longer than that of P and of Q. This leads to a contradiction that P and Q are longest paths in G. □