▶ **Exercise 9-8**

Let $G$ be a self-complementary graph of order $n = 4k$, where $k \geqslant 1$. Let $U = \{v : \deg v \leqslant n/2\}$ and $W = \{v : \deg v \geqslant n/2\}$. Prove that if $|U| = |W|$, then $G$ contains no vertex $v$ such that $\deg v = n/2$.

**Proof.** In the proof, the degrees of a vertex $v$ in $G$ and in $\overline{G}$ are denoted by $\deg_G v$ and $\deg_{\overline{G}} v$, respectively. Suppose that $|W| = p$. Then $|U| = p$. Consider a vertex $v \in W$. Then $\deg_G v \geqslant n/2$. Therefore, $\deg_{\overline{G}} v = (n-1) - \deg_G v \leqslant (n-1) - n/2 = n/2 - 1$. Since there are $p$ vertices $v$ in $G$ with $\deg_G v \geqslant n/2$, it implies that there are $p$ vertices $v$ in $\overline{G}$ with $\deg_{\overline{G}} v \leqslant n/2 - 1$. By the fact that $G$ and $\overline{G}$ are self-complementary, $G$ contains $p$ vertices with degree no more than $n/2 - 1$. Since $|U| = p$, it means that $G$ contains $p$ vertices with degree no more than $n/2$. It follows that there are no vertices $v$ in $G$ with $\deg_G v = n/2$. □